ON THE POSSIBILITY OF THE APPLICATION OF ELECTRONIC DIGITAL COMPUTERS TO ONE OF THE APPROXIMATE METHODS FOR OBTAINING CONFORMAL MAPS

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This note deals with a method of inscribing the largest possible semicircle in a region of definite shape. Knowledge of this method ensures the possibility of the application of electronic digital computers for the conformal mapping onto regions, approximating the upper half-plane. This work is related to that of Lavrent'ev and Shabat [1,2]. Use is made of one of the reflections, executed by the Zhukovskii function.

We consider the simply-connected region G, lying in the upper half-plane and containing $i\infty$. The complement D of the region G with respect to the open upper half-plane will be called, for the sake of brevity, the "residue" [or cut] corresponding to G. The height h of the residue D we will assume to be the largest of the distances of points C (boundary of G with D) from the real axis.

We transform the region G onto the upper half-plane with an accuracy ϵ, i.e. we transform G onto the region G', for the cut D' of which $h' < \epsilon$.

We will use the Zhukovskii function

$$w - a = z - a + \frac{R^2}{z - a} \tag{1}$$

This function maps onto the upper half-plane the region G with residue D, having the form of the semicircle of radius R with center at some point $z = a$ of the real axis. We assume for simplicity that the boundary C is given by a single-valued continuous function $y = y(x)$.

In the region $D = D_1$ we inscribe the semicircle $L = L_1$ with the largest possible radius $R = R_1$ with center at some point $a = a_1$ of the real axis.

The transformation

$$w_1 - a_1 = z - a_1 + \frac{R_1^2}{z - a_1} \tag{2}$$

transforms the region $G = G_1$ into some region G_2 with residue D_2 of height h_2. For this, the residue D_1 decreases all along the boundary. Therefore, $h_2 < h_1$. If $h_2 < \epsilon$, the problem is solved.

If $h_2 > \epsilon$, then in $D = D_2$ we inscribe a semicircle L_2 with the largest possible radius R_2 with center at a_2 on the real axis. The transformation

$$w_2 - a_2 = w_1 - a_2 + \frac{R_2^2}{w_1 - a_2} \tag{3}$$

transforms $G = G_2$ into G_3 with residue D_3 of height h_3. For this the residue D_3 reduces all along. Therefore, $h_3 < h_2$. If, still, $h_3 < \epsilon$, the problem is solved. If $h_3 > \epsilon$, we continue the process until we obtain $h_n < \epsilon$.

Thus, the simple formulas for each successive transformation transfer one or several points of the boundary C to the real axis.

The studied method is specially suitable for the determination of stream lines in hydrodynamics. In fact, after several transformations one may attain the inequality $h < \epsilon$; therefore, one may assume that the stream line with asymptote $v = c$ (at $\pm\infty$ the boundary touches the real axis) lies between the straight lines $v = c$ and $v = c + h$. The inverse transformation $z = z(w)$ maps the strip between $v = c$ and $v = c + h$ onto a curvilinear strip in the z plane. The lower edge of this strip serves as asymptote of the stream line, corresponding in the w plane to the stream line with asymptote $v = c$. The width ϵ_1 of the curvilinear strip at the largest bulge differs little from ϵ. Therefore, the stream lines are determined to an accuracy $\epsilon_1 \approx \epsilon$.

A deficiency of the studied method will be that the calculations must be executed for each point separately. Therefore, it is very important to use for the solution of the problem under consideration the high-speed computer ETsVM. The computations may be done on this computer in two stages:

1) evaluation of the consecutive values of a_i and R_i;

2) evaluation of the successive inverse transformations of the form

$$z = x + iy = \frac{1}{2}\left(w + a + \sqrt{(w - a)^2 - 4R^2}\right) \tag{4}$$

or, more exactly, of the form

$$z = \frac{1}{2}\{u + a + \operatorname{sign}[(u-a)v]\frac{1}{\sqrt{2}}\sqrt{g(u,v)+f(u,v)}\} + \\ + \frac{1}{2}i\{v + \frac{1}{\sqrt{2}}\sqrt{g(u,v)-f(u,v)}\} \tag{5}$$

Here

$$g(u,v) = \sqrt{[(u-a)^2 - v^2 - 4R^2]^2 + 4(u-a)^2v^2},\ f(u,v) = (u-a)^2 - v^2 - 4R^2 \tag{6}$$

Obviously, the construction of the program for the computation of this expression on the ETsVM does not present any major difficulties. We consider the method for obtaining the a_i and R_i.

Let there be given two point sets $\{N'\}$ and $\{N''\}$, each of which contain a finite number of points lying not below the real axis. The point N with abscissa a_0 is the only common point of $\{N'\}$ and $\{N''\}$. It does not lie below other points of the sets above the real axis. The points of the sets have different abscissae. For this the points N' not coinciding with N have abscissae less than a_0, while the points N'' other than N have abscissae larger than a_0.

It may be shown that one can always find on the real axis a unique point A which is equally far away from $\{N'\}$ and $\{N''\}$, if by distance of A from $\{N\}$ we understand the smallest of the distances AN.

It may likewise be proved that the point A may be found by a method of successive approximations involving the following steps. As zero approximation we take A_0, the projection of N onto the real axis. We find the points N_0' of $\{N'\}$ and N_0'' of $\{N''\}$ closest to A_0. Let $A_0N_0' = R_0'$ and $A_0N_0'' = R_0''$. If $R_0' = R_0''$, the problem is solved; if $R_0' \neq R_0''$, then we find the point A_1 on the real axis, lying half way between N_0' and N_0''. For A_1 we find the closest points N_1' of $\{N'\}$ and N_1'' of $\{N''\}$.

Let $A_1N_1' = R_1'$ and $A_1N_1'' = R_1''$. If $R_1' = R_1''$, the problem is solved; if $R_1' \neq R_1''$, we find on the real axis the point A_2, lying at equal distance from N_1' and N_1'', etc. The process ends after a finite number of the described steps.

The abscissa a_1 of the point A_1 follows analytically from the abscissa a_0 of the point A_0, the abscissae x_0' and x_0'' of the points N_0' and N_0'', and the quantities R_0' and R_0'' by use of the formula

$$a_1 = a_0 + \frac{(R_0')^2 - (R_0'')^2}{2(x_0' - x_0'')} \tag{7}$$

Similarly a_2 is found from a_1, x_1', x_1'', R_1' and R_1'', etc.

Taking for $\{N'\}$ some set of points on C' (the part of C which is not to the right of the point N) and for $\{N''\}$ some set of points on C'' (the part of C which is not to the left of N), we will have a method for obtaining the inscribed semicircles and for determining their centers with high accuracy.

After this the program is readily constructed.

1) Give the increment Δx. Obtain $\{x\}$, the set of abscissae of points of the curve C. Store these values in the memory.

2) From $y = y(x)$ find the set of the corresponding ordinates $\{y\}$. Store these values in the memory.

3) Determine h_1, the largest of the obtained ordinates y. Let such an ordinate have the point N_1.

4) Take as zero approximation of the center of the inscribed semicircle the abscissa $a_{1,0}$ of the point N_1 and determine $R_{1,0}'$ and $R_{1,0}''$, the values of which are clear from the preceding work. If $R_{1,0}' = R_{1,0}''$, then $a_1 = a_{1,0}$ and $R_1 = R_{1,0}' = R_{1,0}''$.

5) If $R_{1,0}' \neq R_{1,0}''$, define the position a_1 by the formula

$$a_{1,1} = a_{1,0} + \frac{(R_{1,0}')^2 - (R_{1,0}'')^2}{2\,(x_{1,0}' - x_{1,0}'')} \tag{8}$$

where the meanings of $x_{1,0}'$ and $x_{1,0}''$ are likewise clear from the above.

Then again find $R_{1,1}'$ and $R_{1,1}''$. If these are equal, the problem is solved; if they are not equal, determine a new position of the center. The points 4 and 5 form the first cycle, at the conclusion of which the position of the inscribed circle a_1 is determined. The signal for the final cycle is the equality $R_1 = R_{1,n}' = R_{1,n}''$. The memory gives a_1 and R_1.

6) Complete the calculations for all pairs (x, y) by use of (2).

Then from the found values $v_1 (w_1 = u_1 + iv_1)$ select the largest. Let its ordinate be the point N_2.

If $v_1 \leqslant \epsilon$, the solution has been found; if $v_1 > \epsilon$, we find the new center and radius of the new inscribed semicircle, etc. The points 3, 4, 5, 6 form the second cycle, at the conclusion of which we obtain successive values a_i and R_i $(i = 1, 2)$. The criterion for conclusion of the cycle is the inequality $h = h_n < \epsilon$, where ϵ determines the accuracy of the conformal transformation.

We note that the machine time may be reduced, if one considers on C_i every time only those points which lie on parts of the curve C_i passing through the point C_i and having ends on the real axis.

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